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Understanding the Properties of Arithmetic:

A Prerequisite for Success in Algebra

AS EARLY AS FOURTH GRADE, WE BEGIN introducing students to the properties of arithmetic. The commutative property of addition and the commutative property of multiplication allow us to change the order of the numbers without changing the value of the expression. The associative property of addition and the associative property of multiplication state that regrouping numbers when adding and/or multiplying does not change the solution. The distributive property of multiplication over addition allows us either to find the addend of the numbers, then multiply by the factor, or to first multiply the factor by each addend, then find the sum. Finally, the identity properties of addition and multiplication state that the value of any real number remains unchanged if zero is added to it or if it is multiplied by one, respectively.

To most students, the properties look like a tedious exercise in exploring the obvious. The properties of arithmetic, like algebra, are often defined abstractly. Middle school students need to construct their own understanding of how arithmetic works to arrive at a basic understanding of algebra. Algebra can be considered a generalization of arithmetic, and the properties of arithmetic allow us to stand back from the calculation and look at the generalizations.

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I introduce the properties of arithmetic by first talking about the properties of a seventh grader: starting to grow taller, starting to deal with acne, often worried about who is friends with whom, and so on. The properties of a seventh grader are the descriptions that are true about many of them. The properties of arithmetic are the items that are true about arithmetic. As with seventh graders, the properties of arithmetic are unique to some operations and not to others. For example, although addition and multiplication are commutative, subtraction and division are not.

The Commutative and Associative Properties of Addition and Multiplication

NUMERICALLY SPEAKING, THE COMMUTATIVE and associative properties of addition and multiplication are obvious. When we change the order or the grouping, the sum or product remains the same; for example, $7 + 6 = 6 + 7$ and $7 + (2 + 5) = (7 + 2) + 5$, respectively. Similarly, $7 \cdot 3 = 3 \cdot 7$ and $2 \cdot (3 \cdot 5) = (2 \cdot 3) \cdot 5$.

Many students assume that the commutative and associative properties are interchangeable and may therefore miss the difference between them entirely. Their confusion is supported by the fact that we often use the commutative and associative properties at the same time. When adding $13 + 6 + 7 + 4$, we could rewrite it as $(13 + 7) + (6 + 4) = 20 + 10 = 30$. However, students need to view this as two separate steps: First, rearrange the order of the numbers ($13 + 6 + 7 + 4 = 13 + 7 + 6 + 4$), then change the

grouping to (13 + 7) + (6 + 4). On the other hand, we could start by changing the grouping, then the order, and the grouping again: 13 + 6 + 7 + 4 = 13 + 6 + (7 + 4) = 13 + (7 + 4) + 6 = (13 + 7) + (4 + 6). In other words, first we use one property, then we use the other.

I give my students pictorial images for these properties. When we use the commutative property, we put a term or number or variable on a wagon and move it (see **fig. 1**). I draw that wagon every time we use the commutative property, and students like picturing a wagon moving the numbers around. When the associative property is discussed in my classroom, it involves holding hands. In the expression 7 + (5 + 9), the 5 and 9 are holding hands, and they are excluding the 7. If we rewrite it as (7 + 5) + 9, now the 7 and 5 are holding hands and the 9 is being excluded. As we deal with each expression, I encourage the students to consider whether it is a question of numbers holding hands or of putting a number or variable on a wagon and moving it.

Factoring can prove useful in combination with the associative property of multiplication. How can we multiply 3 • 14 in our heads? We can start by saying 3 • 14 = 3 • (2 • 7). Then we can use the associative property: 3 • (2 • 7) = (3 • 2) • 7 = 6 • 7 = 42. We could do the same thing with 18 • 4: 18 • 4 = (9 • 2) • 4 = 9 • (2 • 4) = 9 • 8 = 72. A thirteen-year-old can see the utility in using the associative property to help with arithmetic.

The associative and identity properties of addition can be used to add numbers with opposite signs: (–8) + 14 = (–8) + (8 + 6). With the associative property, we can then regroup the terms as [(–8) + 8] + 6 = 0 + 6 = 6. The only caveat is choosing the partition of the number that is opposite of (–8), allowing us to end up with the identity property of addition: 0 + *n* = *n*, where *n* is any real number. If we changed (–8) + 14 to (–8) + 5 + 9, it would be no help.

$$7 + 5 = 5 + 7$$

We transport the 7 to the other side of the 5 by putting it on the wagon and moving it.

Fig. 1 A model for the commutative property of addition

An important thing to remember about the commutative and associative properties is that both division and subtraction are neither commutative nor associative. For example,

$$7 \div 6 \neq 6 \div 7,$$

$$\frac{2}{3} \div \frac{5}{6} \neq \frac{5}{6} \div \frac{2}{3},$$

and the expression 7 ÷ 5 ÷ 2 does not have the same value as 7 ÷ (5 ÷ 2). In division, changing the order of the fractions gives the reciprocal of the original problem's answer. In the example above, 5/6 ÷ 2/3 is equal to the reciprocal of the original problem, 2/3 ÷ 5.6.

If we attempt to use the commutative or associative properties with subtraction, we run into similar difficulties; for example, $7 - 5 \neq 5 - 7$ and $3 - 1 - 5 \neq 3 - (1 - 5)$. The results are clearly different. As happens so often, a consideration of what *does not* fit the rule can be just as instructive as what *does* fit the rule. If we change all subtraction to addition of the opposite of the subtrahend and all division to multiplication by the reciprocal of the divisor, then the commutative and associative properties will apply. Doing this helps many students learn to simplify expressions involving subtraction and division correctly.

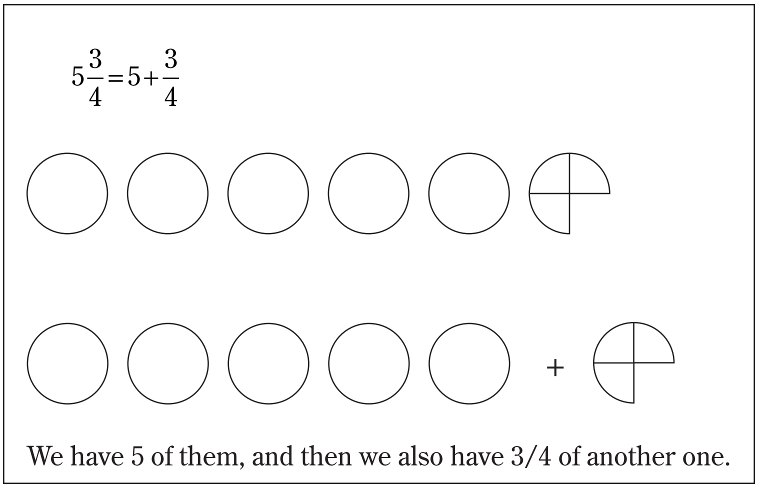

Fig. 2 A model illustrating assumed addition for a mixed number

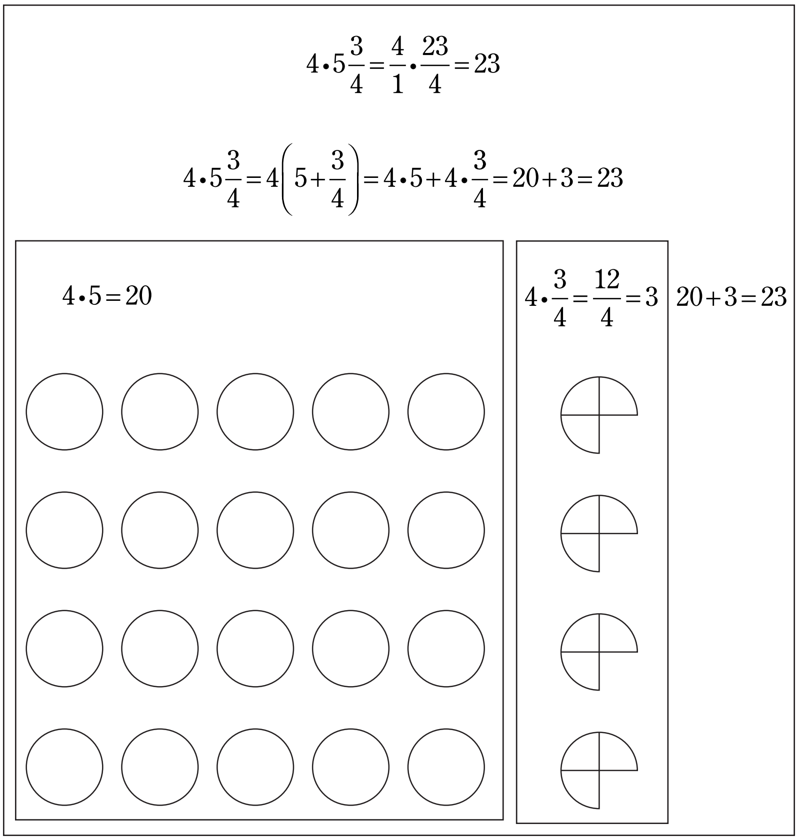

Fig. 3 A model illustrating the distributive property for a whole number times a mixed number

The Distributive Property of Multiplication over Addition

IT IS BOTH EASY AND DANGEROUS TO MISUSE THE distributive property in algebra. Therefore, I teach it in several ways when I am still working at the arithmetic level. If we need to multiply $7 \cdot 99$, we could rewrite that expression as $7(100 - 1)$, and then distribute: $7(100 - 1) = 7 \cdot 100 - 7 \cdot 1 = 700 - 7 = 693$. A frequent error in using the distributive property is forgetting to multiply the factor (the 7 in this case) by the second term (the 1) as well as the first term (the 100). If we were distributing candy bars to all students in the class and stopped after the first child, there would be cries of "Unfair!" Every student wants a candy bar. The expression $700 - 1$ is not equal to $7(100 - 1)$. When we multiply $7 \cdot 99$, we get the same answer that we get for $7(100 - 1) = 700 - 7 = 693$.

Multiplying a whole number by a mixed number is another useful application of the distributive property in arithmetic:

$$4 \cdot 5\frac{3}{4} = 4\left(5 + \frac{3}{4}\right).$$

Some students do not immediately see that

$$5\frac{3}{4} = 5 + \frac{3}{4}$$

but a picture may help (see **fig. 2**). They may also have trouble accepting that the distributive property really works with multiplying mixed numbers but again drawing pictures may help (see **fig. 3**). Using the distributive property sometimes makes the multiplication of improper fractions easier. For example,

$$3\left(5 + \frac{2}{3}\right) = 3 \cdot 5 + 3 \cdot \frac{2}{3} = 15 + 2 = 17.$$

In class, we check our solution by doing the multiplication the traditional way:

$$3 \cdot 5\frac{2}{3} = \frac{3}{1} \cdot \frac{17}{3} = 17$$

The distributive property states that we should get the same answer both ways, and we do.

Some students have a hard time distinguishing between the associative and the distributive properties. Many assume that if they see parentheses, it must mean that the distributive property is being used, but this is not always the case. For example, $7(8 + 4) = 7 \cdot 8 + 7 \cdot 4$ is an application of the distributive property; however, $7(8 \cdot 4)$ is not. The second expression does not imply that we should multiply the 7 by both the 8 and the 4. The commutative and associative properties can be applied to change the order and grouping, but the distributive property does not apply. The expression $7 \cdot 8 \cdot 7 \cdot 4$ is not the same as $7 \cdot 8 \cdot 4$ or $7 \cdot (8 \cdot 4)$. How can a student tell the difference? The distributive property always involves both multiplication and addition or subtraction. The multiplication must be one quantity times a sum or a difference or an implied sum, as in

$$5\frac{3}{4}.$$

If we multiply $7 \cdot 8$ by saying $7(5 + 3)$, then we can distribute and get $35 + 21 = 56$. When we multiplied $7 \cdot 99$, we rewrote 99 as $(100 - 1)$. When we multiplied

$$3 \cdot 5\frac{2}{3},$$

we rewrote

$$5\frac{2}{3}$$

as a sum before we distributed. The distributive property of multiplication over addition means exactly that: We can distribute the multiplication to each addend.

Another mistake that students often make with the properties is trying to apply the commutative or associative property to an expression involving both addition and multiplication. When I ask students to write an explanation of the properties and to show their work, several students give me an example like this: $3 \cdot 5 + 6 = 3 \cdot 6 + 5$. They need to see that the commutative property of multiplication and the commutative property of addition are in fact two separate properties. If they did the actual arithmetic, they would see that $15 + 6$ is not equal to $18 + 5$. The same difficulty can arise with the associative properties. The distributive property, on the other hand, works only when we have either addition or subtraction along with multiplication and then only when the multiplication is distributed over the addition or subtraction—not when the addition is distributed (how would you do that?) over multiplication. I picked up on those errors by asking my students to articulate in writing their understanding of the properties.

Identity Properties of Addition and Multiplication

IN SOLVING EQUATIONS, WE CAN USE THE identity properties of both addition and multiplication ($x + 0 = x$ and $x \cdot 1 = x$). The identity property is closely tied to opposites and reciprocals and, therefore, to the numbers 0 and 1.

When we solve an equation such as $x + 7 = (-3)$, we want to find the value for x. We want to have $x + 0$ equal a number since with the identity property of addition, $x + 0 = x$. We need to replace the 7 with 0. How can we get 0? If we add the opposite of 7 to 7, we get 0; thus, in the equation $x + 7 + (-7) = (-3) + (-7)$, and then $x + 0 = (-10)$ or $x = (-10)$.

To solve an equation like

$$\frac{3}{4}x = 9,$$

we want to find the value of one x. Multiplying both sides of the equation by the reciprocal of 3/4, we get

$$\frac{4}{3} \cdot \frac{3}{4}x = \frac{4}{3} \cdot \frac{9}{1},$$

or $1 \cdot x = 12$. Applying the identity property of multiplication, $1 \cdot x = x$, we have $x = 12$.

Summary

ALGEBRA CAN BE VIEWED AS A GENERALIZATION of arithmetic, and the properties of arithmetic can shed light on this view. With addition or multiplication, we can change the order (the commutative property) of the numbers or variables that we are adding or multiplying, and the result will be the same. We can change the grouping of numbers or variables (the associative property) that we are adding or multiplying, and the result will remain the same. To multiply a number by a sum, we can either (1) add first and then multiply or (2) multiply the number by each addend, then add those products (the distributive property).

The identity properties are used to solve equations. When we want to find the value of $1x$, we multiply the coefficient of x by its reciprocal. To get $x + 0$, we add the opposite of the addend since the sum of any number plus its opposite is 0.

Understanding the properties of arithmetic lays the foundation for students to understand algebra as a generalization of arithmetic. Mastering these concepts in arithmetic helps students succeed in algebra. □