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Algebraic Problem Solving in the Primary Grades

Robert B. Femiano

You may be surprised to learn that most first graders can solve for x in such problems as $2x + 1 = 21$. I know so because they do it in my classroom daily, just not in the abstract form of x's and y's. Even though primary-grade students may lack the formal level of thinking required to "efficiently" solve equations, algebraic reasoning is still possible when approached in less sterile and more practical ways. This article shares teaching experiences that show just that outcome: when an equation is redesigned into a problem-solving story or a logic puzzle or is in some other way wrapped in meaning, even six-year-olds know enough mathematics to solve it.

Equations are mathematically pure by nature; thus, they are devoid of context. This starkness has its advantages but not for concrete learners. To them, manipulating equations is likely a rote—and joyless—procedure because true understanding is absent. Once equations are transformed into concrete problems, however, amazing learning takes place because children can now make sense of the problem; that is, they know what is being asked and can understand the role of an unknown. This understanding allows them to logically generate a solution plan that is based on reasoning rather than on memorized methods. In other words, when students understand the question behind the problem or puzzle, they can use natural thinking skills to formulate solutions, instead of relying on the recall of methods and invariable procedures or algorithms. Students can now bridge their concrete ways of thinking with the abstractness of equations.

This instructional shift away from memorization has several benefits. First, it allows even those students who do not have good skills or strong sup-

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port at home to participate; indeed, the children who are labeled low achievers constantly surprise me with their eagerness and their strong thinking abilities. Similarly, the increased focus on problem solving helps me address the gender equity issue. This benefit may be attributed to the language and "socialness" of problem solving, but in any event, girls appear to be just as motivated as boys and equally successful in finding solutions.

Why Use Problem Solving?

NCTM's *Principles and Standards for School Mathematics* describes problem solving as a "major means for developing mathematical knowledge" (2000, p. 116). A year earlier, in NCTM's yearbook on mathematical reasoning, Schifter made a strong argument for the value of early algebraic thinking in K–6 classrooms as long as the curriculum builds on students' mathematical ideas without separating them from their own sense-making abilities (NCTM 1999).

In my classroom, mathematics is simply problem solving in one form or another, whether the explorations are focused on algebra, geometry, measurement, or data. To foster mathematical thinking, I pose problems that provide numerous reasoning and thinking opportunities. The problems are presented in a friendly and supportive environment that encourages risk taking and in which mistakes are viewed as opportunities for learning. When young children are placed in a failure-free, fear-free environment, they enjoy the puzzlelike nature of algebraic problems. Their enthusiasm also serves to transform mathematics from a passive to an active subject, thereby increasing student participation.

Problem solving is a natural vehicle to help young children uncover vital mathematical relationships and concepts by building on their own knowledge bases. Because problems are frequently solvable in more than one way, students can begin with knowledge they already have and explore from there. Furthermore, problem solving encourages children to rely on their own thinking abilities, thereby helping improve their perceptions of themselves as mathematicians. At a minimum, this confidence boost should serve to inhibit mathematics phobias; better still, perhaps it will be the catalyst for a mathematics-based career. Either way, helping students feel comfortable with mathematics is a worthwhile goal and a lifelong gift.

Three forms of problem solving are used in my classroom—story problems, equations with cards, and balancing problems—each of which is designed to promote some algebraic reasoning skills. These activities help students understand

concepts, such as equality; explore properties, such as the commutative property; learn mathematical language by using such terms as *greater than;* and sharpen computational skills. In this article, each of these problems is represented by equations, which are given solely to clarify the problems for readers. Equations are rarely used in the classroom because they seem unnecessary and too limiting to the thinking of the primary-age child. Equations may be used, however, to record what some children say as they explain their answers. For example, I may say aloud to the group, "He said, 'What do I need to add to 5 to make 9?' which can look like this when it is written down: $\square + 5 = 9$."

Story Problems

Perhaps the most powerful way for elementary school students to solve equations is by putting words to them. Revisiting the opening equation, $2x + 1 = 21$, suppose that the same question was framed in a word problem, such as "Double me, then add 1, and you will get 21. What number am I?" Now, the problem becomes less abstract and much more approachable; at least, it lends itself to

FIGURE 1

How the problem was solved

Equation	Story Problem	Strategy
$x + 5 = 17$	Bobby read some books, then said, "If I read 5 more, I will have read 17." How many books has he read?	Take seventeen counters; separate five for the books that Bobby needs to read, and count what remains as the number that he has already read.
$x - 1/2x = 35$	If you spent half your money but still had 35¢, what 4 coins did you begin with?	Make a pile of thirty-five with base-ten blocks, then add another pile of thirty-five to make equal halves. Then they can find how many they have altogether. Some students might try to find four plastic coins to make 70¢. Depending on the student, I might "require" him or her to find a second answer.
$x + y = 20$; $x - y = 6$	What two numbers give you 20 when added but, when subtracted, give you 6?	Take twenty counters and make two rows, one of which has six more than the other.
$x + x + 3 = 21$	If Jack fetched 3 more pails of water than Jill did, and together, they carried 21 buckets, how many pails did each fetch?	Take twenty-one blocks and make two lines, one of which has three more than the other.

guess-and-check techniques. I have often seen second graders solve this problem by making a table or chart or by first subtracting the 1 from 21, then breaking the 20 into two groups. In the latter solution, students reverse the steps used to state the problem and, in doing so, display a textbook example of how to isolate the unknown to one side of the equation before solving! The same equation could be embodied in many forms, such as "Mom and Dad both made the same number of cookies. Then baby made 1. Altogether, they had 21. How many cookies did Dad make?" I commonly use the students' names in problems; for example, "If Ramona had the same number of pennies in both pockets but then found one more, making a total of 21 cents, how many pennies were in each pocket?"

Here is another typical Algebra I problem: Find y given $x + 3 = 20$ and $x + 5 = y$. Again, in this form, the problem would be too daunting for most twelve-year-olds, but when reformatted into words, even a first grader can reason it out. For example, the problem might read, "Margo will be 20 years old in 3 years, and her brother, Dominic, is 5 years older than Margo. How old is Dominic?" Given counters and encouragement, primary-grade children will be able to master the thinking necessary to find a solution, and along the way, they will continue to explore number properties and concepts and learn mathematical vocabulary in a natural context.

The following examples are designed for the multiage group that I teach, which includes first through third grades. Exposing students to these types of problems, not just once but repeatedly throughout the year, ensures success. Repeated exposures also give me the chance to change the numbers in a way that encourages students to develop more efficient or sophisticated strategies and to formulate generalizations. Other adaptations include using fractions, writing more complex equations, or introducing new terms, such as *consecutive numbers*. In **figure 1,** the story problems are similar to those that my students might find on the chalkboard when they arrive at school. Again, the equation given here is for readers only, whereas the explanation is a sample of how the problem has been solved in my class.

When left on their own, students create many ways to find the answers. At this age, the efficiency of their procedures is not nearly as important as the mathematical validity of their reasoning and calculations. Of course, more mature forms of mathematical thinking exist, but suggesting to these youngsters that they leap past their own understanding in favor of standard symbolism and traditional algorithms seems risky. Because I want to foster mathematical self-confidence, I value reasoning over efficiency. Naturally, if a student is ready to be challenged to think in more complex ways, I attempt to ask questions that will open up the possibility of a shortcut. Like any teacher watching a child draw sixty-nine tallies to represent sixty-nine, I might ask whether the student can think of an easier way to draw sixty-nine. If my question falls on deaf ears, I may try a direct explanation involving tens and ones blocks or encourage the child to look at how someone else in the room does the problem using tens and ones in a drawing. If the child still does not make the change independently, then he or she needs to continue drawing one-to-one correspondences until readiness develops to group in tens. At least the child is not held back by the lack of mastery of one skill when he or she can still think like a mathematician!

When students finish their problems, they are required to explain what they did and justify their reasoning, just as mathematicians do in the real world. This step also serves as a building block in helping my students pass a fourth-grade state mathematics test in which written solutions are required to demonstrate problem-solving abilities. Sometimes, one student's method, even if it is a nonstandard algorithm, may lead to an all-class announcement that someone has uncovered an important idea, such as the fact that the order in which three numbers are added does not affect the sum. Rather than identify this idea as the associative property of addition, it is named after the student, and others are encouraged to try using Ellen's way or Joseph's method.

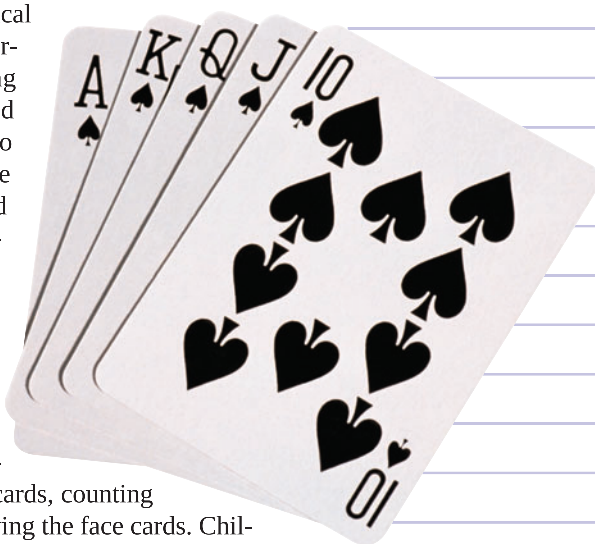
Most often, explanations involve using mathematical tools, such as number lines, base-ten blocks, counting chips, hundred grids, or calculators. Students quickly learn not to ask, "Is this the right answer?" but rather to approach me with "I need help" or "I'm sure I have the right answer!" Always, they must be prepared to tell me how they know; they learn that copying another student's answers is pointless.

When students are not sure where to start, I first ask them to restate the problem, checking their comprehension. I may suggest using lesser numbers in a simplified version of the problem, which helps students focus on the process, not the imposing numerals. I may direct students to look at work from previous days to see whether they can find similarities or may prompt them to solve the problem using charts, tables, or other techniques posted in the room. Sometimes, we might physically act out or draw the problem. As a last resort, students may be encouraged to ask others to show them how they found their solutions. Even if we work through the problem together, students are exposed to fundamental and powerful problem-solving methods that will serve them throughout their school years and beyond.

These types of story problems are also wonderful assessment opportunities; they give me insight into how children are thinking and reveal areas of mathematics in which students need more practice. Furthermore, the possibilities for creating multistep story problems are as limitless as those for equations; similar problems with small twists can be used until pupils show mastery of the concept.

Equations with Cards

In algebra, a solution set is often found by balancing equations, sometimes by adding or subtracting the same amount from both sides. One way to develop the groundwork for such thinking is through the use of missing addends. For example, kindergarteners can typically solve a problem such as the following by counting out the number of blocks needed: "Here are five blocks. How many more do you need to have eight?" The traditional procedure to solve $5 + x = 8$ is to subtract 5 from each side, although this approach is not logical for most children. Furthermore, by finding out the number needed to complement 5 to make 8, students are working with sets and will eventually discover the possibility of subtracting rather than adding.



One way to encourage thinking about missing addends is with a standard deck of playing cards, counting aces as one and removing the face cards. Children are cautioned to consider the number on the card, not the color or suit. This activity connects what children know about concrete objects with a pictorial representation and leads to abstract thinking. When I first introduce the game, I gather a group of students around me. I take the top card and place it faceup on the rug. I then draw a second card, secretly look at it, and place it facedown next to the first card. I tell the students that together, these cards will count to whatever their sum is. I then ask, "Can we find a way to figure out what number is on the hidden card without looking at it?"

FIGURE 2

Four problems involving playing cards

□ + 7 = 10 $(x + 7 = 10)$

□ + □ = 8 + 6 $(2x = 8 + 6)$

The first two cards are the same.

□ + □ + □ = □ + □ $(3x = 2y)$

The first three cards are the same, and the last two cards are the same.

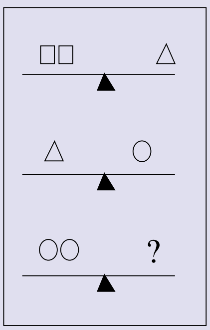
□ + □ + □ + □ = □ + □ + □ + □

All eight cards must be different.

FIGURE 3

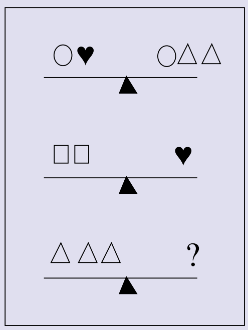
Three balance-pan problems

All scales are balanced. Replace the question mark with one of the choices.



(a) □□
(b) △□
(c) □□□□
(d) ○□

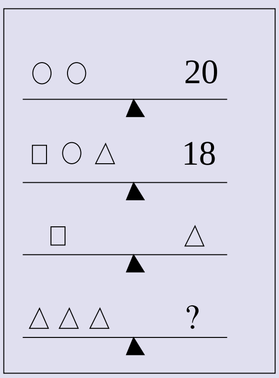
(a)



(a) ○○
(b) ♥□
(c) ♥♥
(d) ○♥
(e) ○□□

(b)

The following variation asks for a number to replace the question mark.



(c)

I also tell students that counting on fingers is acceptable. At first, students seem to use only trial and error to solve these problems, often with no foresight or hindsight. For example, they may try a 5 even if 4 is too large. After a few turns, however, some children's reasoning and number sense becomes apparent. Eventually, the students uncover many concepts, such as equality, identity element, joining sets, and many properties, such as commutative and associative, of number and set theory.

Within a few days of introducing this activity, I begin to make the transition to paper-and-pencil problems. Early in the first-grade year, I write on the chalkboard such problems as those shown in **figure 2.** After students copy the problems into their notebooks, they use their decks of cards to find the solutions. Incidentally, the lack of written words in these problems makes them wonderful activities for nonreaders and non–English speakers to participate equally with their classmates. Note that buying decks of cards with different backings makes cleanup time more manageable for teachers and students.

The last two problems in **figure 2** are unique because they have more than one solution. In algebra, the variable is a fundamental idea, and such problems are one way to expose children to it. Another way would be to say, "I am more than 1 but less than 5. What number could I be?"

Another card game that I use quite often is called "salute" (Kamii 1988). This game requires three players and a deck of cards with the face cards removed. Shuffle and divide the deck into two equal piles. The third player is the referee. The two opposing players each draw a card from their facedown piles and, without looking at the card, place it on their foreheads with the number facing out, in a salute fashion. The referee adds the numbers and announces the total. Each player, seeing what the opponent has, tries to be the first to shout the number on his or her card. Because players get only one try, they have to think carefully. Tie answers, determined by the referee, result in cards' being returned to the bottom of the players' decks. The winner collects both cards, and the game continues until one player wins all the cards.

Balancing Problems

Simultaneous equations are basic to algebra and involve dual unknowns that are dependent on each other, such as the example cited previously in the brother-sister age problem. One way to solve these equations is to use substitutions, in which the value of one unknown, expressed in terms of the other unknown, replaces the same variable in the second equation. For instance, if $x + 1 = y$, then $x + 1$ could be substituted in place of y in the second equation. This substitution eliminates the second variable and leaves an equation with only one unknown, which can be solved easily. To use substitution, however, a student's sense of equality must be somewhat developed. To this end, I devised balance problems, such as the following: If ten blue weights at 1 gram each balance a yellow weight of 10 grams and ten blue weights balance two red weights, then will two reds balance one yellow? Before working with these problems, students have opportunities to use pan balances.

Sometimes, these problems are solved by eliminating the same amount from both sides; again, this approach can be explored on the balance pans. If four reds + one blue balances two yellows + one blue, then removing the blue from both sides of the equation will leave it unchanged. Removing something from only one side would obviously result in an imbalance, which would not be mathematically acceptable but might be a good launching point for discussing inequalities. See **figure 3** for a few examples used with my second and third graders.

Conclusion

Algebraic problem solving has proven to be an invaluable tool in helping children develop mathematical and logical thinking skills. It not only strengthens conceptual understanding but also provides many other benefits, from reducing mathematics anxiety to increasing participation levels. Furthermore, allowing children to explore and invent mathematical procedures using their own thinking empowers them; they come to see themselves as competent, confident mathematicians.

This shift in teaching, from telling to allowing discovery, has made a profound difference in how my students view mathematics. Practically speaking, however, with large classes and extremely diverse populations, the task of changing curricula and methods without support is not easy. For example, I bring in a parent every day to help with mathematics time. I hope, however, that this article offers some encouragement for change in how we teach, because even if our steps in this direction are small, such as using a problem-of-the-week format, the benefits are enormous: children smiling, even laughing, during mathematics time.

References

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