Fostering Relational Thinking while Negotiating

the Meaning of the Equals Sign

NCTM's *Principles and Standards for School Mathematics* (2000) recommends that algebraic thinking be developed beginning in the elementary grades. Teachers can accomplish this goal while teaching other important components of the elementary curriculum by helping children attend to patterns, relations, and properties of operations in all mathematics activities. True-false and open number sentences provide a good context for achieving this goal while also revealing information about students' knowledge of operations and properties. In this article, we relate an experience in which we used number sentences to begin to develop students' algebraic thinking.

Integral to children's work in algebra is an understanding of the equals sign. Unfortunately, children often have serious misconceptions about the meaning of the equals sign (Behr, Erlwanger, and Nichols 1980; Falkner, Levi, and Carpenter 1999; Saenz-Ludlow and Walgamuth 1998). Children tend to perceive the equals sign as a stimulus for an answer and react negatively to number sentences that challenge their conceptions of the equals sign. For example, children often change 3 + 2 = 2 + 3 to 3 + 2 + 2 + 3 = 10 and 8 + 4 = __ + 5 to 8 + 4 = 12 + 5 =17 for expressing a string of operations. Frequently, older students continue to have difficulty using the equals sign correctly (Mevarech and Yitschak 1983; Byers and Herscovics 1977).

Understanding the equals sign is associated with what Carpenter, Franke, and Levi (2003) call relational thinking. When students use relational thinking, they can solve number sentences by focusing on the relationships between the numbers in the equation instead of performing all the computations. For example, in the sentence 27 + 48 – 48 = __, students might recognize that adding and then subtracting 48 will leave 27 unaffected, therefore avoiding computation. This particular problem does not require a broad understanding of the equals sign, because all the computation takes place on the left of the sign. Other equations, such as 8 + 4 = __ + 5, can be solved only if students have a broad understanding of the equals sign. Students can solve this sentence using relational thinking by noticing that 5 is 1 more than 4, so the unknown number has to be 1 less than 8. This more sophisticated approach to solving number sentences has been observed in some elementary-grade students (Carpenter, Franke, and Levi 2003; Koehler 2004), but little evidence is available about the sequence of instructional activities that led to this thinking or how readily students developed it.

Working with third graders, we set out to explore the following questions:

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- How do students' conceptions of the equals sign evolve when considering and discussing varied true-false number sentences?
- Do students develop relational thinking while the class negotiates the meaning of the equals sign?
- Do students retain the new interpretation of the equals sign over time?

We worked with eighteen students over five sessions that took place during the students' regular school time (see **table 1** for schedule of activities). The class was ethnically and linguistically diverse. Five students spoke a second language, and two of them had significant difficulty understanding English. During the preceding months, we, as guest teachers, had worked with the class on a weekly basis doing a variety of mathematics activities. The classroom teacher was always present and sometimes collaborated with us in helping the students.

Session 1: What do students understand about the use of the equals sign?

We began by giving the students an individual assessment with some open number sentences to determine their understanding of the equals sign. All but the second sentence could easily be solved using relational thinking by looking across the equals sign to compare the numbers. None involved computation that would be challenging for most third-grade children. The students had difficulty interpreting these open number sentences. No student gave more than one correct answer, revealing that they all held the misconception that the equals sign is a stimulus for an answer (see **table 2**).

After the students handed in their answers, we discussed two of the problems. All the students thought that the answer to 8 + 4 = ___ + 5 was 12. We told them that "mathematicians" would disagree. When we spotlighted the presence of 5 on the right side, they suggested the answer 17, attained by adding all the numbers. Then, when we said that mathematicians would still disagree, a student proposed that we modify the sequence to 5 + 8 + 4 = 12. Finally, we explained that mathematicians use the equals sign to show that the whole expression on one side is equal to the whole expression on the other side. A student then gave the answer 7. Our discussion of the second problem, 14 + ___ = 13 + 4, reflected a similar pattern in student thinking.

The use of the equals sign in these sentences seemed unnatural to the students. We discussed the fact that they probably had never seen sentences like this before. One student specifically asked why the equals sign was in the middle of the sentences. Through this discussion, we emphasized mathematicians' interpretation of the equals sign to establish it as a convention.

Table 1

Schedule and Organization of the Lessons

Session	1	2	3	4	5
Month	November	February	February	March	May
Number of students in class	13	15	18	18	15
Session activities	Written assessment Brief discussion	Written assessment Discussion Written activity Brief discussion	Discussion Written assessment	Discussion	Written assessment
Number sentences used	Open number sentences	True-false number sentences	True-false number sentences	True-false number sentences	Open number sentences

Session 2: How do students' conceptions of the equals sign evolve when considering and discussing varied true-false number sentences? Two months later, we assessed the students to determine how many of them had adopted the mathematicians' interpretation of the equals sign. We chose true-false sentences to challenge students to first analyze problems rather than immediately jump to computation. When students saw an operation sign, they wanted to complete that operation even before looking at the right-hand side of the equals sign. Having them consider true-false sentences was a good way to force them to look at the whole sentences while challenging their conceptions of the equals sign. Our resource for these sentences was Carpenter, Franke, and Levi (2003). We asked students to correct false number sentences to provide us with further information about their interpretation of the equals sign.

Table 3 shows the number of correct responses for each sentence. Three students responded appropriately to all the sentences except the first two, and when correcting false sentences they wrote sentences of the form $a + b = c + d$. We inferred that these three students had remembered and understood our previous discussion. Six students had begun accepting "backward" sentences (for example, 10 = 4 + 6) but not sentences in the form $a + b = c + d$. The assessment showed that two-thirds of the class continued to have a misconception about the equals sign despite our previous discussion.

We had assumed that pointing out their misconceptions might be enough to change the students' minds about the equals sign. Clearly it was not. The students needed opportunities to come to their own understanding of mathematicians' uses of the equals sign by sharing their thinking with one another. We used the assessment questions to spark discussion and began by asking students the meaning of the equals sign. Students' responses included "It is like if you have a scale and you have to put the same amount of both, on each side, for it to be equal" and "It means equal to, the same amount." We continued to negotiate the meaning of the equals sign with the students by discussing opposite opinions about each sentence and different ways of fixing the false sentences. For example, when discussing the sentence 2 + 2 + 2 = 3 + 3, some students affirmed it was true, and one explained, "It is true because 2 + 2 + 2 does equal 6 and so does 3 + 3." Other students said that they thought it was false and explained, "I thought it should be 2 + 3 = 5" or "I thought it was false because it has the equals sign in the middle."

During the discussion a student said that "34 = 34 + 12 is false because 34 + 12 would be more than 34." The student did not report adding the numbers to decide whether the sentence was true or false but instead had compared both quantities, 34 and 34 + 12, showing the beginning of relational thinking. We could see the students struggling to make sense of some of the sentences. Such comments as "They want to trick you!" indicated that we had successfully created dissonance, from which learning might result.

Table 2

Analysis of Student Responses to Open Number Sentences

Sentences on Day 1	Correct Answer	Most Common Incorrect Answer	Other Answers
8 + 4 = __ + 5	7 (0)	12 (13)	None
__ = 25 – 12	13 (3)	7 (2)	5, 17, 18, 20, 25, 35
14 + __ = 13 + 4	3 (2)	1 (5)	–1, 0, 3, 17, 31
12 + 7 = 7 + __	12 (3)	26 (3)	0, 3, 5, 6, 7
13 – 7 = __ – 6	12 (0)	6 (11)	3, 7
__ + 4 = 5 + 7	8 (0)	1 (9)	2, 3, 12

Note: The numbers in parentheses indicate the number of students who gave each response (N = 13).

Table 3

Analysis of Student Responses to False Number Sentences

Sentences on Day 2	Correct Answer	Number of Students Responding		
		True	False	No Answer
3 = 3	T	5	9	1
7 = 12	F	2	10	3
10 = 4 + 6	T	9	5	1
2 + 2 + 2 = 3 + 3	T	5	9	1
34 = 34 + 12	F	2	11	2
99 + 4 = 4 + 9	F	3	8	4
37 + 14 = 38 + 13	T	5	6	4

Note: The number of students who responded correctly is highlighted (N = 15).

Figure 1

Student equations to produce true sentences

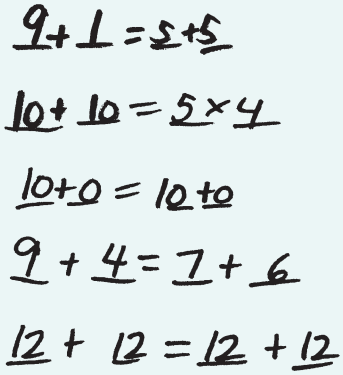
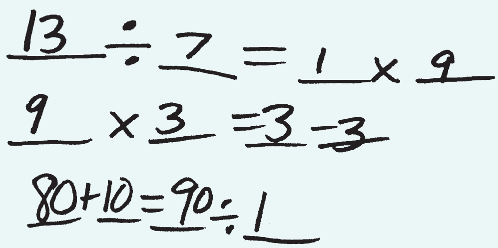


Figure 2

Student equations using the identity property



To further assess and stimulate students' conception of the equals sign, we asked them to write true sentences of the following forms:

$$\_\_ + \_\_ = \_\_ + \_\_,$$
$$\_\_ - \_\_ = \_\_ - \_\_, \text{ or}$$
$$\_\_ + \_\_ = \_\_ - \_\_.$$

These templates might confuse students who interpret the underline as a variable and assume that the same number must go in each blank. This format did not cause a problem for our students. Writing number sentences was a good activity for helping them clarify and consolidate their understanding. All but two of the students were able to write sentences of this form, although we helped four of them get started because they were writing sentences of the form $a + b = c$.

In this activity, students could generate sentences that were more difficult or less difficult, depending on their own choice. The students used multiplication and division as well as addition and subtraction in their sentences (see **figs. 1** and **2**). In many instances, as in the examples in **figure 2,** students multiplied or divided by 1; in other instances, they added 0. Students' choices of these easy ways to generate number sentences gave us insight into their knowledge of the identity properties. Although such sentences showed understanding of the equals sign, they did not show relational thinking. In contrast, some students' sentences did show relational thinking. A few sentences took the form $a + b = (a - 1) + (b + 1)$, as in $51 + 51 = 50 + 52$. One student used relational thinking to decompose addends in changing 90 to a string of addends (see **fig. 3**).

One of the students showed a tendency to write the answer of the operation in the middle of the sentence, thereby separating both sides (see **fig. 4**). We also observed this pattern in another student's work on day 2. This format seems to show a necessity to have the "answer" in the equation and could be the bridge between the more familiar form $a \pm b = c$ and the less familiar form $a \pm b = c \pm d$.

Session 3: Do students develop relational thinking while the class negotiates the meaning of the equals sign?

Because we wanted to promote relational thinking and because we realized that the students' understanding of the equals sign was still fragile, two weeks later we planned a discussion of some true-false sentences (see **fig. 5**), some of them previously written by the students. In an attempt to promote the use and verbalization of relational thinking, we asked the students whether they could solve the sentences without doing the arithmetic. The students were engaged in sharing their thinking.

When discussing the first sentence, $20 + 20 = 20 + 20$, most students claimed, "It is true because they are the same numbers" and "You don't need to write the answer." One student stated, however, "It is false because the equals sign is in the middle"; this comment reminded us that not all students had adopted the mathematicians' interpretation of the equals sign. In the sentence $7 + 15 = 100 + 100$, all the students were sure it was false. They explained, "It is false because $7 + 15$ is small and $100 + 100$ is 200" and "$7 + 15$ is not even 100." Students also evidenced steps toward relational thinking in their comments about other sentences, showing the kind of thinking we were hoping to foster. They also

Figure 3

Student equations produced by decomposing addends

$200+200=400-0$ $\quad$ $201+300=500+1$

$90+200=200+10+10+20+30+20$

explained that "[51 + 51 = 50 + 52] is true because if you take the 1 from the 51 to the other 51, you get 50 + 52" and that "[15 + 2 = 15 + 3] is false because 3 is bigger than 2."

We observed a significant growth in students' understanding. They used the equals sign with a broader interpretation when correcting the false number sentences, proposing such sentences as 7 + 193 = 100 + 100, 10 – 7 = 7 – 4, and 15 + 3 = 15 + 3.

The discussion of the last sentence, 3 + 3 + 3 = 9 + 2 = 11, was especially interesting. We tried to challenge the students' understanding of the equals sign by playing the devil's advocate. One student claimed, "I think it is false because 3 + 3 + 3 = 9 and 9 + 2 = 11," to which we responded, "Isn't that what it said there?" Regardless of our attempts to cause confusion about the incorrectness of the sentence, some students' knowledge of the equals sign was now firm enough to maintain their opinion about the statement's incorrectness, and they defended their opinion throughout the discussion. One said, "3 + 3 + 3 does not equal 11." Other students became confused, saying, "I am not sure. . . . It is in part true, and it also seems false" and "It is true because 3 + 3 + 3 = 9 and 9 + 2 = 11." Because a difference of opinion persisted, we finally explained that mathematicians would say it was false because 3 + 3 + 3 does not equal 9 + 2.

Our final activity on this day was to assess students' understanding of the equals sign with some written questions (see **fig. 6**). Twelve of the eighteen students solved at least five of the six items correctly. Three students continued to have the misconception of the equals sign as a stimulus to give an answer but showed acceptance of "backward" sentences ($c = a \pm b$). Another three students did not successfully perform the assessment activity. As a result of our discussion and activity on day 2 and this discussion on day 3, nine more students seemed to have constructed a broader understanding of the equals sign.

Figure 4

Student equations introducing the answer at midsentence

$10+11=21=20+1$ $\quad$ $10\times1=10=10+0$

$2\times2=4=2+2$

$20+10=30=30-0=30$

Session 4: Once students correctly interpret the equals sign, do they use relational thinking to evaluate number sentences?

Two weeks later we had a discussion to elicit relational thinking and consolidate students' understanding of the equals sign. We discussed true-false sentences (see **fig. 7**), and more students verbalized relational thinking. In all but one of the sentences (34 + 28 = 30 + 20 + 4 + 8), students gave explanations based on relational thinking, stating, for example, that "[27 + 48 – 48 = 27] is true because there is a plus 48 and a minus 48 [and] that's going to be zero"; or that "[103 + 205 = 105 + 203] is true because 5 + 3 = 8 and there are two eights matching, so they are both the same"; or that "[12 – 7 = 13 – 8] is true because they added 1 to the 7 and they added 1 to the 12." In these examples students did not compute to determine their response; yet some students gave justifications based on the computation of the operations on each side, showing that they were not exclusively using relational thinking.

During this discussion seven of the students' contributions to the discussion displayed relational thinking. Only one comment evidenced some remaining misconception about the use of the equals sign. From the previous days we knew that this student tended to consider the equals sign as a stimulus for an answer, and during the discussions

Figure 5

True-false number sentences for the discussion on day 3

20 + 20 = 20 + 20	6 – 6 = 1 – 1
10 × 10 = 100 = 90 + 10	10 – 7 = 10 – 4
7 + 15 = 100 + 100	51 + 51 = 50 + 52
12 + 11 = 11 + 12	5 + 1 = 7 – 1
15 + 2 = 15 + 3	3 + 3 + 3 = 9 + 2 = 11
3 × 5 = 15 ÷ 1	

Figure 6

Assessment activity on day 3

1. Fill the blank with a number that makes the number sentence true.
   5 + 1 = ___ + 2
   4 + ___ = 2 + 2 + 2
   ___ + 0 = 30 – 10
2. Decide whether the number sentence is true or false.
   9 = 5 + 4 T F
   3 + 7 = 10 + 6 T F
   8 = 8 T F
3. Write a number sentence that is true.

he did not seem to become aware of a broader conception.

Session 5: Do students retain the new interpretation of the equals sign over time?

To determine the durability of students' understanding of the equals sign, two months later we gave students an assessment in which six open number sentences were presented in the same format as those on the initial assessment but with different numbers. We also included an additional item with larger numbers. Twelve of the fifteen students correctly answered five of the seven sentences, showing that they understood the meaning of the equals sign. Another two students exhibited the misconception of the equals sign as a stimulus to give an answer but showed acceptance of "backward" sentences ($c = a \pm b$). One student did not solve the assessment correctly and did not show a clear conception of the equals sign.

We included one additional item, 238 + 49 = ___ + 40 + 9, to assess the use of relational thinking, asking for an explanation. Four students did not have time to complete the problem. Seven of the fifteen students successfully solved the problem, four of whom provided clear explanations reflective of relational thinking. One wrote, "Because 40 + 9 = 49, then you add 238, then it makes the same answer." Another wrote, "You split 49 in 40 and 9 and it's the same." Other successful students provided explanations that were more difficult to interpret. We believed that these students used relational thinking, because they recorded no computation on their papers. In addition to these seven students who were successful with the 238 + 49 item, four other students verbalized their relational thinking on day 4. Thus, eleven students had begun to use relational thinking in their analysis of the equations.

We observed that two of the students regressed in their understanding of the equals sign between day 3 and day 5, answering incorrectly at least five of the seven sentences on day 5 by considering the equals sign as a stimulus for an answer. Three other students improved in their understanding of the equals sign from day 3 to day 5, correctly solving most of the sentences in the assessment on day 5. The rest of the students showed a stable performance from day 3 to day 5.

Conclusions

We successfully helped the students broaden their conceptions of the equals sign through the different tasks assigned. Students' conceptions evolved from perceiving the equals sign as a "stimulus for an answer," to accepting its use in "backward" sentences, to understanding it as an indicator of equality of expressions. The variety of sentences that we presented challenged students' understanding and forced them to think in new ways about the symbol. Asking the students to write their own sentences was particularly beneficial in helping students assimilate new information and consolidate their broadened conceptions, because they had to use the sign themselves rather than evaluate someone else's use of it.

We were only partially successful in initiating relational thinking. Getting students to step back and look at the whole equation is a challenge because they are used to focusing on computation and proceeding from left to right as they read number sentences. Understanding equations requires considering the whole sentence, beginning at the equals sign and then looking to both sides of it. We observed that some sentences, such as 11 + 3 = 4 + ___, forced the students to step back and try to make sense of the whole sentence because they knew that 11 + 3 = 14. In the future we plan to use more sentences of this form to stimulate relational thinking. Discussion

Figure 7

True-false sentences for the discussion on day 4

37 + 23 = 142
27 + 48 – 48 = 27
34 + 28 = 30 + 20 + 4 + 8
76 = 50 – 14
$4 \times 5 = 5 + 5 + 5 + 4$
20 + 15 = 20 + 10 + 5
103 + 205 = 105 + 203
12 – 7 = 13 – 8

is crucial to fostering relational thinking, because it removes the emphasis from computing answers and places it on noticing patterns.

Because all the students in our class held the "stimulus for an answer" interpretation of the equals sign, we chose to introduce the mathematical convention for it. Our initial explanation was sufficient for only a few of the students to adopt the convention. More students acquired an understanding of the equals sign after they engaged in more discussion and had more experience with it. We hypothesize that they benefited a great deal from hearing how their peers explained their interpretations. A few students went back and forth between their original conception and their newer conception, suggesting that developing a robust understanding of the equals sign can take considerable time.

We are convinced by our foray into true-false and open number sentences that they are a fruitful way to initiate algebraic thinking. The beauty of these tasks is that they have multiple entry points. In solving them, students can use computation or can begin to consider relationships between numbers and operations. These multiple approaches provide necessary ingredients for successful discussion. We conclude that negotiating new interpretations of the equals sign by exposing students to such unfamiliar number sentences is an important first step on the road to students' attainment of algebraic thinking.

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