

Fostering Mathematical Thinking and Problem Solving: *The Teacher's Role*

Effective mathematical problem solvers are flexible and fluent thinkers. They are confident in their use of knowledge and processes. They are willing to take on a challenge and persevere in their quest to make sense of a situation and solve a problem. They are curious, seek patterns and connections, and are reflective in their thinking. These characteristics are not limited to problem solvers in mathematics or even in schools; they are characteristics desired for all individuals in both their professional and personal lives (National Research Council [NRC] 1985; NCTM 1989, 2000; Steen 1990). These characteristics help individuals not only learn new things more easily but also make sense of their existing knowledge. Problem-solving habits of mind prepare individuals for real problems—situations requiring effort and thought, lacking an immediately obvious strategy or solution. To develop problem-solving habits of mind, students need experiences work-

ing with situations that they “problematize with the goal of understanding and developing solution methods that make sense for them” (Hiebert et al. 1996, p. 19).

Current mathematics education reforms at both the state and the national level suggest that students should have such learning opportunities and recommend that increased attention be given to problem solving in all areas of the curriculum (NCTM 1989, 2000). However, simply asking teachers to increase the attention given to problem solving does not ensure a focus on fostering students’ understanding and sense making through problem solving.

This article focuses on two teachers’ implementation of a patterning task and discusses ways in which the teachers guide and manage the discussion about the task. Also described are important considerations for teachers who want to foster their students’ mathematical thinking and problem solving.

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Two Approaches to Problem-Solving Instruction

The vignettes in **figures 1** and **2** present a picture of instruction. Although both teachers are responding to the same recommendations for a focus on



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and thinking; and finally they individually have an opportunity to reflect on what has been shared, explore whether they can create general statements or formulas, and generally construct their own meaning of the information.

The dilemma is that the instructional approaches advocated by many commercially available problem-solving resources and curricula encourage teachers to train their students with “how to” approaches to problem solving, much like those used by the teacher in classroom A. They develop students as *problem performers*, students focusing on an end or completion of a problem. Teachers using these materials may—

- present problems that can be solved without much cognitive effort;
- supply or expect the use of a specific heuristic or strategy for solving a problem; or
- provide their students with specific formats for their problem response or write-up (e.g., restate the problem, explain your thinking, check your work).

When students experience these learning opportunities, they develop a narrowly defined view of mathematics and problem solving. These instructional approaches can leave students dependent on prescribed processes and unable to readily face problems without an immediately obvious strategy. The short-term goal of developing students who complete the problem (*problem performers*) may be attained, but the more important long-term goal of developing flexible and fluent mathematical thinkers (*problem solvers*) may not be.

Comparing Classrooms A and B: The Teachers’ Actions and Decisions

When comparing classroom A (**fig. 1**) with classroom B (**fig. 2**), we can see some similarities. The teachers pose similar problems, and they ask students for ideas about the problem. We see evidence of students’ interacting with one another by building on one another’s ideas. We can also see some differences, differences that can be significant when viewing problem solving beyond prescribed formats or strategies. Although the teacher in classroom A may be preparing students to respond to a set of questions aligned with a particular way of approaching problem solving (e.g., restating the problem, explain-

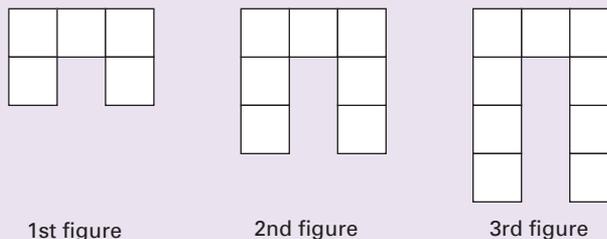
problem solving, they offer students very different learning opportunities.

The focus in classroom A (**fig. 1**) is on finding a single strategy to obtain the answer to the problem. This pursuit is followed by writing up the response to the problem according to specified criteria: restating, explaining, and verifying. This teacher’s view of problem solving, and perhaps mathematics, is that only one way exists to correctly solve the problem and that only one way can be used to write up a response to a problem. By contrast, the instructional focus in classroom B (**fig. 2**) is on exploring the relationships within the problem, sharing the possibilities, and considering how this thinking may extend to any figure in the pattern. This teacher’s view of problem solving is that multiple ways exist to solve a problem and that problem solving is a process of exploring, developing methods, discussing methods, and generalizing results. Students in classroom B at first have a chance to explore the problem individually; they then share their ideas

Figure 1

Vignette—Classroom A

How many tiles are in the 25th figure of this pattern?



- Teacher 1:* Today we are going to practice problem solving. I'd like you to work on answering this question. [Teacher places problem on the overhead projector.] You need to be sure to restate the question, explain your process, and check your work. What is the question asking us to do?
- Alex:* Find out how many tiles are in the twenty-fifth figure.
- Teacher 1:* Good. How are we going to find out?
- Micah:* Just keep adding two tiles until we get to the twenty-fifth figure.
- Alex:* We need to add two tiles twenty-three more times.
- Teacher 1:* OK. What strategy is this? Look at the poster. [Teacher reminds the students of a poster that lists the various problem-solving strategies. Various students make guesses about which strategy Micah and Alex have suggested, finally deciding that they used a "Look for a Pattern" strategy.]
- Teacher 1:* Great. On your paper explain how you'll find the answer, and don't forget to check your work.

ing how you found the answer, checking your work), is she preparing students to problematize and make sense of situations and invent solution methods? Do the students give evidence that they are thinking flexibly and fluently about the problem and considering alternative strategies? The students in classroom A are not engaged in the same level of thinking and reflection as the students in classroom B. The learning opportunities in these classrooms do not yield the same educational outcomes. The main differences and their corresponding outcomes are described more fully in the following sections.

The problem

The problem posed in classroom A is straightforward, asking how many tiles are in the twenty-fifth figure of this pattern. The problem yields only one correct solution. The problem posed in classroom B differs in that the task itself encourages exploration of the pattern and naturally yields a generalization from students. No question is presented; instead, students are asked to investigate and report, thus asking and answering questions that are of interest to them.

Eliciting student thinking

The questions posed by the teacher in classroom A—"What is the question asking us to do?" "How are we going to find out?" "What strategy is this?"—yield responses that simply answer the question asked and communicate little about how the student decided on the solution or what the student sees in the model. The questions do not require deep student thinking; rather, they funnel toward a particular set of information that the teacher wants the students to include in their solution. In classroom B the questions—"What do you notice about this pattern?" "Would you come and show how you see that?" "Did anyone see it in a different way?"—suggest that this teacher values both the process and the product, inviting all students to engage in the conversation. In addition to sharing a solution, the students share their reasoning (how they see their approach in the model), build on others' ideas, consider more than one approach, and make sense of one another's approaches. The teacher expects students to problematize and make sense of the situation and then provide a rationale for what they discover.

Reflection and sense making

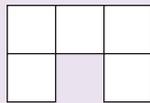
The students in classroom A are not encouraged to take time for reflection and sense making. They move from the task to a focus on the product. The students in classroom B experience several points of teacher-prompted reflection and sense making. The interchange opens with students' individually exploring this open-ended problem in their journals. It next moves through several students' sharing observations and informal generalizations. The discussion gives evi-

dence that students are actively considering the ideas of others, as in the instances of Sage and Omar. Students, having already shared their ideas, build on their peers' ideas as the conversation unfolds. Finally, the teacher asks the students to individually consider all these ideas and then to formulate some thoughts about more formal generalizations and proof. The students in both classroom A and classroom B invent solution strategies; however, in classroom B, students also are asked to explore multiple strategies and

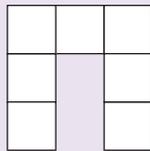
Figure 2

Vignette—Classroom B

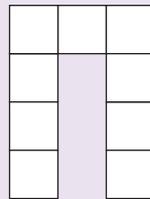
Investigate and report all you can about this pattern.



1st figure



2nd figure



3rd figure

Teacher 2: Please take out your problem-solving journal and begin work on this problem. [*The teacher places the problem on the overhead projector. The teacher circulates as students begin work. After seven minutes, the teacher asks a question.*] What do you notice about this pattern?

Sage: It looks like a table whose legs are getting longer and longer.

Griffin: The legs are getting longer by one tile . . .

Sage: . . . and there are two tiles added for each figure.

Xiao: There are always three tiles on the top of the figure.

Omar: The legs are always the same as the figure number.

Teacher 2: Would you come and show how you see that?

Omar: [*Walks to the overhead projector*] See, the legs are always the same as the figure number. In the first figure there is one [tile] in each leg; in the second there is two; in the third, three, and so on.

Elise: [*Walks to the overhead*] I saw the legs as the whole side. So the legs are longer than the figure number, and there is one [tile] in the middle instead of three on top. In the first figure, I see two [tiles] in each leg, three in the second, four in the third, and there's always one left to count in the middle.

Teacher 2: Did anyone see it in a different way?

Taylor: I saw a "three by" rectangle with empty spaces in the middle. [*Walks to the overhead projector while talking*] There are always three tiles in the dimension [*points along the top dimension of each figure*]. There is always one more than the figure on this dimension—first there is two, then three, then four—so in the tenth figure there would be eleven; it's a three-by-eleven with ten empty spaces in the middle.

Omar: Oh! So the empty space is the same as the figure number.

Sage: You could move that outside leg of the table over by the other leg and make a "two by" rectangle with one extra tile sticking out. [*Walks to the overhead projector and shows how the right leg can slide to the left to fill the empty space*]

Teacher 2: I can see you all have done some good thinking about this. I'd like you to take some private think time now and record your thoughts about some different ways that you might find the total number of tiles in any figure that is in this pattern. Also, how might you convince someone else that you are correct?

analyze these strategies. They are encouraged to consider how they might integrate what they learn from the discussion into their thinking about the problem and ultimately how the various ideas might lead to generalizations.

Cohen (1988) suggests that in a traditional classroom, knowledge is objective and stable, consisting of facts, laws, and procedures that are true, whereas in a reformed classroom, knowledge is emergent, uncertain, and subject to revision. The teacher in classroom A, although attempting to engage the students in a discussion about the problem, asks questions that do not elicit much discussion. Once she hears the response she is looking for from students, she quickly turns the focus to the procedure that the students need to follow in writing up a response to the problem. By contrast, in classroom B the teacher guides the inquiry through posing open-ended tasks, encouraging reflection, and asking questions that draw out students' thinking; generally, she helps students learn how to construct knowledge through interacting with the problem (Cohen 1988) and one another (Leinhardt 1992).

Connecting Reform-based Goals with Teacher Beliefs and Actions

Principles and Standards for School Mathematics (NCTM 2000) suggests that problem-solving instruction should enable students to build new knowledge, solve problems that arise in mathematics and beyond, apply and adapt a wide variety of strategies, and monitor and reflect on the process. Teachers' actions and decisions related to these expectations, as seen in classrooms A and B, often vary and are influenced by the teacher's beliefs about mathematics, problem solving, students' abilities, and so on.

In a recent study exploring influences on the teaching of mathematical problem solving (Rigelman 2002), the four focus teachers identified their main goals for problem-solving instruction. These goals are to help students develop (a) flexible understanding of mathematical concepts; (b) confidence and eagerness to approach unknown situations; (c) metacognitive skills; (d) oral and written communication skills; and (e) acceptance and exploration of multiple solution strategies. Summarized in **table 1** are the relationships among these exemplary teachers'

goals in teaching problem solving, their beliefs regarding the results of having students engage in problem solving, and their specific actions that support problem-solving behaviors in their classrooms. Reading across the table from left to right suggests a link among (a) the teacher's goals for instruction (e.g., fostering students' confidence and eagerness to approach an unknown situation); (b) the teacher's beliefs about problem-solving opportunities (e.g., allowing students to observe, invent, conjecture, generalize); and (c) the teacher's actions (e.g., encouraging reasoning and proof). In the example of the four focus teachers, strong connections are observed among their reform-based goals, their beliefs about the benefits for students, and their actions, emphases, and decisions in the classroom.

Implications for Teachers and Students

In classrooms in which problem-solving instruction focuses on the previously listed goals, the role of the teacher and the role of the students change. Instead of focusing on helping the students "find an answer," the teacher is prepared to see where the students' observations and questions may take them. Instead of providing solution strategies, the teacher encourages multiple approaches and allows time for communication and reflection about those strategies. Instead of expecting specific responses, the teacher is ready to ask questions that uncover students' thinking and press for the students' reasoning behind the process.

These expectations on the part of the teacher affect the students' role, as well. As students engage in problem solving to learn mathematics content, they engage in the work of mathematicians. They explore the problem, and from this exploration they develop models and methods of thinking about the problem. From these models and methods, students develop their reasoning and prove their thinking to be reasonable and valid. Also, in this process, they discuss their reasoning and their solution(s). **Figure 3** shows this cyclical model representing the mathematical problem-solving process. The model is a circle, indicating that the process is unending and that the actions, although somewhat sequential, may not be brought to completion before engaging in the next action; some actions may occur simultaneously, and not all students will be at the same

Table 1

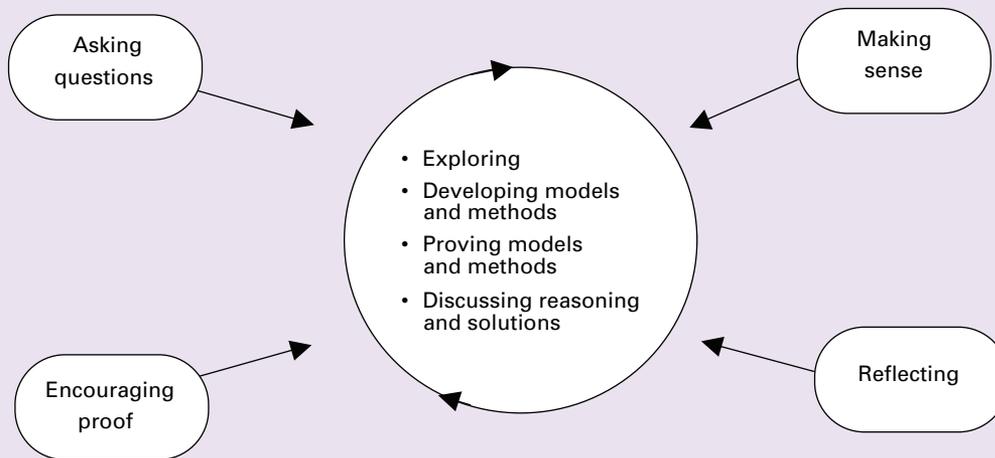
Interrelationships among Teachers' Goals, Beliefs, and Actions Regarding Problems and Problem Solving

Teacher's Goals for Problem-solving Instruction	Teacher's Beliefs about Opportunities Afforded by Problem Solving*	Teacher's Related Actions
<ul style="list-style-type: none"> • Help students develop flexible understanding of mathematical concepts • Foster students' confidence and eagerness to approach an unknown situation • Help students develop their metacognitive skills • Help students develop their oral and written communication skills • Foster students' acceptance and exploration of multiple solution strategies 	<ul style="list-style-type: none"> • Students learn concepts and apply existing understanding • Students observe, invent, conjecture, and generalize • Students see multiple approaches 	<ul style="list-style-type: none"> • Poses problems and asks questions • Encourages reasoning and proof • Encourages reflection • Has students discuss strategies, share ideas, and collaborate toward solutions • Encourages multiple approaches

*Indicates that teachers also choose problems on the basis of these criteria.
 Source: Adapted from Rigelman (2002, p. 171)

Figure 3

Mathematical problem-solving process



Adapted from Rigelman (2002, p. 184)

place in the process as they engage in problem solving. For example, students may discuss their reasoning while they are developing their models.

Additionally, teachers also actively engage in this process; they ask questions to elicit students' thinking, encourage proof, make sense of mul-

multiple approaches, and reflect for the purpose of making informed instructional decisions.

Conclusion

The purpose of this article is to raise questions about what is required to foster mathematical thinking and problem solving in our students. Because we want our students to possess the following habits of mind, we may also need to possess those same habits of mind with regard to instructing our students in problem solving. These habits of mind are embraced by teachers who—

- think flexibly and fluently rather than focus instruction on a particular way of thinking;
- confidently use mathematical knowledge and processes rather than specific strategies based on problem types;
- willingly persevere and make sense of a situation rather than expect students to follow the process that makes sense to them; and
- engage in reflective thinking rather than rote follow a procedure without taking the time to consider what is happening and why.

When teachers carefully choose tasks that require students to engage in mathematical thinking and problem solving, then draw out students' thinking through their questioning, and finally encourage reflection and sense making, their students make significant gains in mathematical understanding and attain higher levels of achievement.

Action Research Ideas

Review your curriculum materials and other problem-solving resources with the following questions in mind:

- Will students learn something new through engaging with the task?
- In what ways do these resources prompt or direct students' thinking?
- Does the support that the materials provide lead students toward a particular strategy for solving the problem or a particular format for the response?
- How might you open up the task so that students have opportunities to deeply explore the mathematics embedded in the task (e.g., to make observations, conjectures, and generalizations)?
- How might you help students make sense of

correct strategies and formats for solving the problem?

After completing a problem with your class, reflect on the focus of the discussion and the written communication. Consider not only your answers to the questions but also the evidence you have gathered about the extent to which these behaviors were present.

- Were students' thinking and reasoning guiding the discussion or evident in the written work? Why or why not?
- Were all students actively engaged in the discussion (individually solving, sharing and comparing their solution strategies, listening attentively, building on one another's ideas, synthesizing the results)?
- Were students sharing both how and why their methods work? Were students able to convince others of the correctness of their solution?

We recommend that you reflect on these questions periodically, making note of areas in which you have improved and setting new goals for yourself and your students.

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